

Note on the asymptotic approximation of a double integral with an angular spectrum representation

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Abstract. In this note, we are concerned with the asymptotic approximation of a class of double integrals which can be represented as an angular spectrum superposition. These double integrals typically appear in electromagnetic scattering problems. Based on the synthetic manipulation of the method of steepest descent path, approximate expressions of the double integrals are derived in terms of the leading term of the contribution to the asymptotic expansions.

1 Introduction

The methods of steepest descent path (SDP) and stationary phase are commonly useful for the asymptotic evaluation of the complicated integrals in electromagnetics particularly in the far zone Lang-LM. For the one-dimensional case, it seems no restrictions need to be enforced on the integrands to apply these classic asymptotic methods, although sometimes the construction of the SDP is difficult as also is the establishment of the uniform asymptotic expansions in the neighborhoods of the singularities of the integrands close to the saddle points Wong, Van. But for the asymptotic approximation of double

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integrals

$$I(\lambda) = \iint_D \varphi(x, y) \exp [i\lambda h(x, y)] dx dy, \quad \lambda \rightarrow \infty, \quad (1)$$

the stationary phase method is generally used provided that the integrands have certain restricted forms, i.e., the phase functions $h(x, y)$ are real-valued BW, Wong, BMM, Jone1. When the complex-valued phase functions are concerned, asymptotic expansions of the double integrals are still possible to be obtained if the integrands are smooth sufficiently and vanish on the boundary of the integration domain D . In this case, the dominant contribution to the asymptotic expansions may come from the critical points of the first kind. However, if the phase function is complex-valued in general but not differentiable everywhere in the domain, some additional work may be necessary to account for the contribution of the boundary stationary points to the asymptotic expansions of the double integrals Wong1-Jone2. These boundary stationary points are certainly located in the vicinity of regions wherein the phase function is not differentiable, i.e., branch points and singularities. Therefore a curve of stationary points may be formed on the boundaries of the sub-domains $D_j \subset D$ which are disjoint with each other by the “deleted” neighborhoods of nondifferentiation. In addition, because the phase functions now are complex-valued, approximation of the Laplace-type integrals may need to be manipulated on some sub-domains to accomplish the classic asymptotic evaluation of these double integrals Wong,, Jone2 [†].

In this note we present a simpler way to establish the asymptotic approximation of a class of double integrals which are derived from an angular spectrum representation of the field-related entities. After departing from the SDP method operated in a synthetic

[†]The boundary stationary points are typically the critical points of the second kind. If there is a curve of stationary points formed on the boundary of the domain D , we know that the leading term of the contribution to the asymptotic expansion of the double integral in (1) is $O(\lambda^{-1})$ as $\lambda \rightarrow \infty$. Similarly, for the asymptotic expansion of the Laplace-type integral $I(\lambda) = \iint_D \varphi(x, y) \exp [-\lambda h(x, y)] dx dy$, if the stationary points are the critical points of the first kind, then the leading term of $I(\lambda)$ is also $O(\lambda^{-1})$ as $\lambda \rightarrow +\infty$. For more details, see Wong,, Jone2.

fashion, the approximation of these double integrals is achieved by using the contour deformation to confine the integration on a small domain defined by two local SDPs. The saddle points along each SDP are employed to construct the truncated Taylor expansion of the complex-valued phase function which is differentiable on the mapped domain. The double integral is then asymptotically evaluated by the lowest order in the far-zone limit. A condition for the asymptotic approximation to be valid is also briefly commented.

2 Theory

To begin with let us suppose the vector $\mathbf{r} = (x, y, z)$ defined on the upper half space $\mathbb{R}_+^3 = \{z > 0\}$ and a double integral $G(\mathbf{r})$ generally having the form

$$G(\mathbf{r}) = \iint_{\mathbb{R}^2} f(k_x, k_y) \exp[i(k_x x + k_y y + k_z z)] dk_x dk_y, \quad (2)$$

where $k_z = \sqrt{k_0^2 - k_x^2 - k_y^2}$ is defined on the top Riemann surface such that $\text{Im}(k_z) \geq 0$ for $k_x^2 + k_y^2 \in \mathbb{R}$. Generally speaking, $G(\mathbf{r})$ represents a class of functions that have the physical importance in reducing a field-related entity into an angular spectrum superposition. It is reasonably assumed that $|G(\mathbf{r})| < \infty$ is well defined by the integrand belonging to $L^1(\mathbb{R}^2)$. In most cases, we may also assume that $f(k_x, k_y)$ in (2) is independent of \mathbf{r} and has no singularities on the complex domain $\Omega = \mathfrak{D} \times \mathbb{R}^c \cup \mathbb{R}^c \times \mathfrak{D}$, where \mathfrak{D} is a complex neighborhood of the real axis, and $\mathbb{R}^c = \mathbb{C} \setminus \mathbb{R}$.

According to Fubini's theorem, $G(\mathbf{r})$ in (2) could be rewritten as

$$G(\mathbf{r}) = \int_{-\infty}^{\infty} \exp(ik_x x) dk_x \int_{-\infty}^{\infty} f(k_x, k_y) \exp[i(k_y y + k_z z)] dk_y. \quad (3)$$

By denoting $g(k_x) = \int_{-\infty}^{\infty} f(k_x, k_y) \exp[i(k_y y + k_z z)] dk_y$, we know that $g(k_x) \in L^1(\mathbb{R})$ almost everywhere. Indeed, $g(k_x)$ may have isolated singularities which depend on $f(k_x, k_y)$. However, because $f(k_x, k_y)$ is assumed to be nonsingular in Ω , it is clear that the isolated singularities of $g(k_x)$, if exist, are only real-valued. Conventionally $g(k_x)$ can be expanded

in the form $g(k_x) = \sum_{n=0}^{\infty} a_n (k_x - s_n)^{\gamma_n-1}$ in the vicinity of the singular point $s_n \in \mathbb{R}$. Since $g(k_x) \in L^1(\mathbb{R})$, it seems that those singularities $\{s_n\}$ are not likely to be poles, so that $\gamma_n > 0$. On the other hand, we can reform $g(k_x)$ as the multiplication of two parts

$$g(k_x) = \tilde{f}(k_x) \exp(ik_{\rho x} \rho_x), \quad (4)$$

where $\rho_x = \sqrt{r^2 - x^2}$, $r = |\mathbf{r}|$ and $k_{\rho x} = \sqrt{k_0^2 - k_x^2}$ with $\text{Im}(k_{\rho x}) \geq 0$ for real k_x . By substituting a23 into a22 and noting that $\tilde{f}(k_x)$ has no poles, it is always feasible to reduce $G(\mathbf{r})$ in a22 into a contour integral along the SDP of the phase function. Therefore, we obtain synthetically the first asymptotic approximation

$$\begin{aligned} G(\mathbf{r}) &= \int_{-\infty}^{\infty} \tilde{f}(k_x) \exp[i(k_x x + k_{\rho x} \rho_x)] dk_x \\ &= \int_{\text{SDP}_1} \tilde{f}(k_x) \exp[ik_0 r \cos(\alpha_x - \psi_x)] dk_x \\ &\approx \int_{\text{SDP}_1^{\text{loc}}} \tilde{f}(k_x) \exp[ik_0 r \cos(\alpha_x - \psi_x)] dk_x, \end{aligned} \quad (5)$$

as $k_0 r \rightarrow \infty$. In a24, $k_x = k_0 \cos \alpha_x$, $\psi_x = \cos^{-1}(\frac{x}{r})$ and $\text{SDP}_1 : (-\infty, \infty) \mapsto \mathbb{C}$ is determined by the parametrization of α_x (or k_x) through the equation $\text{Re}[\cos(\alpha_x - \psi_x) - 1] = 0$. $\text{SDP}_1^{\text{loc}}$ denotes the local path of the SDP_1 that passes through the saddle point $k_{xs} = k_0 \cos \psi_x$.

On account of a22–a24, by using Fubini's theorem again we obtain

$$\begin{aligned} G(\mathbf{r}) &\approx \int_{-\infty}^{\infty} \exp(ik_y y) dk_y \int_{\text{SDP}_1^{\text{loc}}} f(k_x, k_y) \exp[i(k_x x + k_z z)] dk_x \\ &= \int_{-\infty}^{\infty} \tilde{f}(k_y) \exp[i(k_y y + k_{\rho y} \rho_y)] dk_y, \end{aligned} \quad (6)$$

where the function $\tilde{f}(k_y)$ is defined by the equation

$$\int_{\text{SDP}_1^{\text{loc}}} f(k_x, k_y) \exp[i(k_x x + k_z z)] dk_x = \tilde{f}(k_y) \exp(ik_{\rho y} \rho_y). \quad (7)$$

Here $\rho_y = \sqrt{r^2 - y^2}$ and $k_{\rho y} = \sqrt{k_0^2 - k_y^2}$ with $\text{Im}(k_{\rho y}) \geq 0$ for real k_y . Actually the definition of $\tilde{f}(k_y)$ in ¤26 is very similar to that of $\tilde{f}(k_x)$ in ¤23. Following the same approach as illustrated in ¤22–¤24, we can obtain the second asymptotic approximation of $G(\mathbf{r})$ as follows:

$$\begin{aligned} G(\mathbf{r}) &\approx \int_{\text{SDP}_2} \tilde{f}(k_y) \exp[ik_0 r \cos(\alpha_y - \psi_y)] dk_y \\ &\approx \int_{\text{SDP}_2^{\text{loc}}} \tilde{f}(k_y) \exp[ik_0 r \cos(\alpha_y - \psi_y)] dk_y. \end{aligned} \quad (8)$$

In ¤27, $k_y = k_0 \cos \alpha_y$, $\psi_y = \cos^{-1}(\frac{y}{r})$ and $\text{SDP}_2 : (-\infty, \infty) \mapsto \mathbb{C}$ is determined by the parametrization of α_y (or k_y) through the equation $\text{Re}[\cos(\alpha_y - \psi_y) - 1] = 0$. $\text{SDP}_2^{\text{loc}}$ denotes the local path of the SDP_2 that passes through the saddle point $k_{ys} = k_0 \cos \psi_y$.

It seems that $G(\mathbf{r})$ can be asymptotically approximated by either ¤24 or ¤27. Clearly, the asymptotic expression of $G(\mathbf{r})$ is strongly dependent on the properties of $\tilde{f}(k_x)$ and $\tilde{f}(k_y)$ in the fact that the leading contribution to $G(\mathbf{r})$ would be significantly influenced or even dominated by the singularities of $\tilde{f}(k_x)$ (or $\tilde{f}(k_y)$) that are close to the saddle point k_{xs} (or k_{ys}) Wong, Wong2. On the other hand, the synthetic equations ¤23 and ¤26 imply that both $\tilde{f}(k_x)$ and $\tilde{f}(k_y)$ have been modulated by the larger parameter $k_0 r$. Therefore, the asymptotic expansion of $G(r)$ cannot be obtained in closed-form, unless the asymptotic expansion of $\tilde{f}(k_x)$ (or $\tilde{f}(k_y)$) is achieved beforehand.

However, the asymptotic approximation of $G(\mathbf{r})$ can still be accomplished by restoring $G(\mathbf{r})$ with a double integral representation. In fact, due to ¤25–¤27, the approximation of $G(\mathbf{r})$ is now reduced to a double integral

$$G(\mathbf{r}) \approx \iint_{D'} f(k_x, k_y) \exp[i(k_x x + k_y y + k_z z)] dk_x dk_y, \quad (9)$$

where the domain $\mathbb{C} \supset D' := \text{SDP}_1^{\text{loc}} \times \text{SDP}_2^{\text{loc}}$ is located in a neighborhood of the critical point (k_{xs}, k_{ys}) . The local pathes of the SDPs can be represented parametrically by the

real-valued ξ and η as

$$k_x(\xi) = k_{xs} + k_{zs}(1 - i)\xi, \quad (10)$$

$$k_y(\eta) = k_{ys} + k_{zs}(1 - i)\eta, \quad (11)$$

where $k_{zs} = \sqrt{k_0^2 - k_{xs}^2 - k_{ys}^2}$. Therefore D' can be viewed a continuous map from the local domain $\xi \times \eta \subset \mathbb{R}^2$ by means of [a29](#) and [a210](#).

Let $U(\xi, \eta) = (k_0 r)^{-1}(k_x x + k_y y + k_z z) - 1$ be defined on D' ; then $G(\mathbf{r})$ in [a28](#) can be written in the form

$$\begin{aligned} G(\mathbf{r}) &\approx \exp(ik_0 r) \iint_{D'} f(k_x, k_y) \exp[ik_0 r U(\xi, \eta)] d\xi d\eta. \\ &\approx f(k_{xs}, k_{ys}) \exp(ik_0 r) \iint_{D'} \exp[ik_0 r U(\xi, \eta)] d\xi d\eta, \end{aligned} \quad (12)$$

by utilizing the approximation $f(k_x, k_y) \approx f(k_{xs}, k_{ys})$ on D' . Indeed, to be more accurate, we need to express $f(k_x, k_y)$ in term of the Taylor expansion by assuming $f(k_x, k_y)$ is analytic in a neighborhood of the critical point. There will be no additional difficulty in working out the asymptotic approximation of $G(\mathbf{r})$ in [a211](#) when $f(k_x, k_y)$ can be represented by its Taylor expansion. Furthermore, if $U(\xi, \eta)$ or $k_z = \sqrt{k_0^2 - k_x^2 - k_y^2}$ is differentiable on the domain D' , then we can approximate $U(\xi, \eta)$ by its truncated Taylor expansion on the critical point $(\xi, \eta) = (0, 0)$. In fact, $k_z = \sqrt{k_z^2}$ is holomorphic on the complex domain $\mathbb{C} \supset D_2 := \{k_z^2\}$ continuously mapped from D' as shown in Figure 1, simply because D_2 is located on a Riemann surface that neither crosses the branch cut nor encloses the branch point $(0, 0)$. Therefore $U(\xi, \eta)$ is analytic on D' in accordance with the chain rule of differentiation. Then $U(\xi, \eta)$ is approximated by the Taylor expansion at $(0, 0)$ truncated to the second order as:

$$U(\xi, \eta) \approx U_T(\xi, \eta) = i(a\xi^2 + b\eta^2 - 2c\xi\eta), \quad (\xi, \eta) \in D', \quad (13)$$

by recalling $\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial \eta} = 0$ at $(0, 0)$. Here

$$a = 1 - \frac{y^2}{r^2}, \quad b = 1 - \frac{x^2}{r^2}, \quad c = \frac{xy}{r^2}, \quad (14)$$

are real-valued constants.

After substituting a212 into a211, we then derive the asymptotic approximation ($k_0 r \rightarrow \infty$)

$$\begin{aligned} G(\mathbf{r}) &\approx f(k_{xs}, k_{ys}) \exp(ik_0 r) \iint_{D'} \exp[-k_0 r(a\xi^2 + b\eta^2 - 2c\xi\eta)] d\xi d\eta \\ &\approx f(k_{xs}, k_{ys}) \exp(ik_0 r) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-k_0 r(a\xi^2 + b\eta^2 - 2c\xi\eta)] d\xi d\eta, \end{aligned} \quad (15)$$

which is the leading term of the factor $(k_0 r)^{-1}$ of the contribution to the asymptotic expansion of $G(\mathbf{r})$. In fact, the integral on the right side of a214 can be easily calculated by eliminating the cross-product term $\xi\eta$ through a linear transformation.

It is worthy to note that the approximation of $U(\xi, \eta)$ by its truncated Taylor expansion a212 can be reasonably achieved only under the condition that $\theta = \frac{z}{r} > 0$ is not too small. In fact the Taylor expansion of $U(\xi, \eta)$ at (0,0) takes the form $U(\xi, \eta) = \sum_{|\beta|=2}^{\infty} u_{\beta} \underline{\xi}^{|\beta|-2} [1 + P_{\beta}(\underline{\xi})]$ where $\underline{\xi}^{\beta} = (\xi^{\beta_1}, \eta^{\beta_2})$ and $\beta = (\beta_1, \beta_2)$ is a multiindex such that $|\beta| = \beta_1 + \beta_2$. If the coefficient u_{β} is kept as a factor of $O(1)$, then the function $P_{\beta}(\underline{\xi}) \sim O(|\underline{\xi}|/\theta)^{|\beta|-2}$ could be achieved at some β for any $|\beta| \geq 2$. After considering the approximation of a214, $|\underline{\xi}|^2 \sim O(1/k_0 r) \in D'$ is at least guaranteed. Therefore, $\theta > O\left(\sqrt{\frac{1}{k_0 r}}\right)$ is needed for an appropriate approximation of $U(\xi, \eta)$ by a212 on the domain D' . In other words, the asymptotic approximation of $G(\mathbf{r})$ by a214 becomes reasonable only when $\theta > \theta_0$ is necessarily satisfied, where $\theta_0 = \sqrt{\frac{1}{k_0 r}}$.

3 Conclusion

In this note, we are concerned with the asymptotic approximation of a class of double integrals which can be represented as an angular spectrum superposition. These double integrals are typically useful for the calculation of field-related problems in physics and electromagnetics. A simpler way is found to make the asymptotic approximation feasible

and reasonable by applying the steepest descent path method on the double integral in a synthetic fashion. After reduction to a local integration on a small domain in which the complex-valued phase function is differentiable, the asymptotic approximation of the double integral is finally derived in terms of the leading term of the contribution to the asymptotic expansion. The validity of the asymptotic approximation is only under the condition that $\frac{z}{r} > \theta_0$ is satisfied, where $\theta_0 = \sqrt{\frac{1}{k_0 r}} > 0$.

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Figure Captions

Figure 1. Schematic of the continuous map $k_z^2 : D'(\xi, \eta) \mapsto D_2(k_z^2)$.

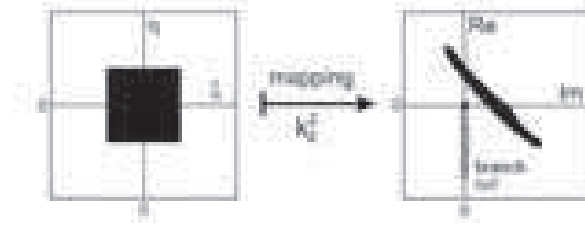


Figure 1: Schematic of the continuous map $k_z^2 : D'(\xi, \eta) \mapsto D_2(k_z^2)$.